Simple Reasoning with Time-Dependent Propositions

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Abstract

Simple practical reasoning with propositions whose truth values depend on time is a matter of logical engineering. One needs an expressive language in which simple inferences are productive. Here's one approach, along with some algorithms for implementing it. We also consider reified and non-reified logics, and show that, contrary to a claim of Bacchus *et. al.*, a reified logic is more appropriate than its non-reified equivalent, even for Boolean logic, when time references are interpreted as union-of-convex intervals.

1 Introduction

Many simple propositions may be true at one time and false at another. Any database containing data from the real world must consider such a possibility, and there are various ways of timestamping entries in relation tables (i.e., atomic formulae) to reflect temporal dependencies. However, not everything can easily be configured as relations amongst atomic formulae. Sometimes it is best to maintain small databases and use rapid inferences. What sort of rapid inference mechanism might be worthwhile for temporally-dependent information?

The computational sciences consider temporality in Artificial Intelligence (especially planning), and in describing and analysing distributed, concurrent, and real-time systems. Systems for temporal reasoning have been developed, the tension between expressiveness and tractability remarked, and different solutions proposed, corresponding to different concerns. In AI, temporal logics have been considered widely for theoretical purposes, effective reasoners for particular domains have been built [GS93, GS9x, LR92, LR93, vB92a] and yet well-developed applications are comparatively lacking (except, oddly, as an alternative to generally successful combinatorial methods for scheduling [ZDD+92]. We also note the Time Map Manager of Dean [DM87, Dea89], extended at Honeywell [SCB92]). Sometimes one can discover important features of reasoning by looking again at very simple cases.

We consider simple reasoning amongst Boolean formulae which are evaluated as true and false on intervals of time. First, we consider only *temporally irreflective* propositions, such as

'*John is wearing a green coat*' that make no obvious reference to times other than the time of assertion (Section 1.1). Second, we look for simple rules of the form ($\bigwedge$ *hypotheses* $\Rightarrow$ *conclusion*), in which each basic formula asserts that a proposition is true over a period of time, or not. Our reasons are in Section 1.2.

We argue the case for evaluating propositions on *union-of-convex* intervals, intervals which consist of separate, convex interval components (Sections 2 and 2.1). We use the term *union-of-convex interval* in this paper for intervals with a finite number of components only. We use the TUS (Time Unit System) notation for union-of-convex intervals over the rational numbers from [Lad86a, Lad87a] (Sections 2.3, 4, and 4.1), and use it to describe algorithms for the useful interval-combining operations of *conglomeration* (similar to union) and *intersection* (Section 3.1, Figures 1 and 2). Conglomeration is a partial operation on convex intervals, but is total on union-of-convex intervals. Intersection is also partial on union-of-convex intervals if the empty interval is not available.

We propose some principles of inference for temporally-irreflective proposition/interval pairs in Section 3.2. Temporally irreflective propositions fulfil the conditions of *downward persistence* (DP), that if a proposition is true on an interval, then it's true on any subinterval of that interval; and *limited upward persistence* (UP), that if a proposition is true on an interval I, also on another J, then it's true on the conglomeration of those intervals. We take these as fundamental and consider what other principles must be used for Boolean reasoning with such propositions. A collection of basic principles appear in Table 1 and the derivable principles in Table 2.

Finally, we compare reified and non-reified formalisms. We find that a non-reified logic such as proposed by Bacchus, Tennenberg and Koomen (hereafter called the BTK logic) is less appropriate than a reified logic for our setup (Section 5). (Bacchus *et al.* had proposed their logic independent of considerations about the application or the temporal data structure [BTK91]). A translation of the principles of Section 3.2 into BTK logic yields the non-reified versions of (DP) and (UP), plus an extended form of Modus Ponens. However, some of the other principles, when translated into a propositional non-reified form, yield false statements! This information is summarised in Tables 3 and 4. There are no obvious quantifier-free replacements for the reified principles that are false in BTK translation, and they must therefore be dropped from a non-reified formulation, weakening its expressive power. We interpret this phenomenon as showing that the reified form is a more expressive carrier of the principles of propositional reasoning evaluated on union-of-convex intervals. This contrasts with the conclusion of Bacchus *et al.* for propositions evaluated on points or convex intervals (Section 5).

1.1 Two Comments on Temporal Irreflectivity

Temporally-irreflective propositions are intuitively those that in a natural language would be expressed in the present tense with no temporal adverbs or other such explicit reference to time other than the present. Some (e.g., [Dow79, All83]) argue that interpretation of the continuous present tense, which we take as an example of temporally-irreflectivity, requires a semantics over intervals. Propositions such as '*Fred Smith is employed by Jones Cat-Cleaning Industries, Inc.*' are temporally-irreflective: temporal qualifiers such as '*yesterday*', '*next week*' do not occur. Intuitively, the truth value of a temporally-irreflective proposition depends on time, but not on indirect relative temporal references to specific times, or to '*the past*' or '*the future*' of the time of assertion.

We noted that temporally-irreflective propositions satisfy (DP) and (UP). However, there are propositions which are intuitively not temporally irreflective that also satisfy (DP) and (UP) – for example, propositions in some versions of linear tense logic. These propositions may contain temporal operators (adverbs) which ensure that the truth of the subsentence qualified by the adverb must be evaluated over other distinct times in the future (or the past). However, other propositional temporal logics with more complex adverbs, such as those of [HS91], appear not to satisfy (DP) and (UP).

If p and q are temporally irreflective, it seems that Boolean combinations of them should be temporally irreflective also. Quantifiers lead to a few puzzles: if a temporally irreflective atomic predicate P is true of an object whose name is a, but which exists at a different period of time from another object whose name is b, then to evaluate $(\forall x : P)$ on an interval on which neither object exists throughout, how do we handle the range of the quantifier? Does it range over both a and b (and other objects), or only over a and b during the subperiod of their existence? And if a does not exist over a subperiod, what's the truth value of $P(a)$ on that subperiod?

We avoid treating either the issue of exactly characterising temporal irreflectivity or that of quantification in this paper: our proposals concern Boolean combinations of atomic propositions that satisfy (DP) and (UP).

1.2 Constraints

We motivate here the hypotheses constraining the inquiry. We were led to this work by considering the enhancement of an ATMS with temporal qualification to the truth of its propositions, as in [BCMH94]. This application has four constraining features:

1. A focus on syntax. Information concerning the relations between propositions, intervals and truth must be reduced to syntactic information and syntactic inference from axioms;
2. Inference is quantifier-free. An ATMS handles only Boolean combinations of propositions, and inference follows so-called forward-chaining (i.e., Modus Ponens using axioms which are conditionals);
3. Basic propositions considered are typically temporally irreflective;
4. It must be straightforwardly implementable.

The first feature suggests a search for a way of saying '*proposition p is true on interval i*' within the object-language of formulas and inference, rather than as a meta-statement about a logical system. The second suggests that we should be trying to find axioms of the form ($\bigwedge$ *hypotheses* $\Rightarrow$ *conclusion*); the third that we can assume (DP) and (UP). The fourth constraint is assured by our choice of structure: for example, [Lad87b, Ch. 9] desscribes one way to implement some union-of-convex interval calculations, and a version of conglomeration was used in the PMA system [DJL88]; consistency-maintenance and management techniques are described in [FdK93]. Designing a temporal ATMS itself is beyond the scope of this paper.

The second constraint precludes translations involving quantification. This will become important in Section 5: accurate translation of 'reified' principles into the non-reified BTK language is possible using quantifiers in the BTK language, but appears to be *only* possible using quantification.

2 Time

What structure is needed to represent real-world temporal information? Suppose the CEO of your company has two jackets, one red with orange polka dots and the other orange with red polka dots. You naturally want to keep a daily record of his dress, in order to explain how it contributed either to your company's meteoric rise or to its complete misreading of the market, but in either case to make millions with your book on it. The CEO wears his red jacket on Mondays, Wednesdays and Fridays, and his orange jacket on Tuesdays and Thursdays, or the other way round, depending on the week. According to some mathematicians, and some AI researchers, his jacket is red at all points in the set of points from the first point at which he starts work on Monday until the point at which he leaves work on Monday. According to others who simplify, it's red *from* the first point .... *to* the point at which he leaves. However, being ordinary mortals we just want to say it's red for the workday on Monday. But we also want to say that this is true on other days too, with gaps of orange in between.

The first simplification leads to an ontology of convex intervals of time as the temporal reference of truth values of propositions and the second to an ontology of objects which are unions of separated convex intervals, which we call *union-of-convex* intervals.

There's a difference between what's in an ontology, and what's represented in the underlying logic utilising this ontology. For example, Shoham [Sho87] uses two 'points' to specify the temporal validity of propositions which are true between these two points. Two points define a *convex interval*, as in [All83]. We would say that his representation uses points, but his ontology is that of convex intervals. It doesn't matter much in this case since the two are logically interdefinable [Lad87c] and a well-known view of Quine [Qui61, Qui69, Qui70] shows that we can consider ourselves equally ontologically committed to two classes of things when they are logically interdefinable (although of course it matters what sort of 'logical interdefinability' one considers, and what sorts of mathematical objects one believes exist).

Truth over explicit convex intervals and related structures has been studied in [Ham71, Hum79, McD82, All83, All84, BL89, vB92b], modal logics using intervals in [HS91, Ven92], and [Lad86a, Lad86b, MAK91, MSK93] have studied propositions and reasoning over union-of-convex intervals; [Lig90, Lig91] has further studied the mathematics of representations of convex intervals as sequences of points of varying length.

2.1 Choosing a Representation

If we want to evaluate the truth of propositions over union-of-convex intervals, we must choose a representation for the union-of-convex intervals we care about. Standard logical languages use predicate or function symbols with a fixed number of places. An Allen-Hayes-type convex interval theory corresponds to a theory using pairs of points (see [Lad87c]). Thus, one can use either two point arguments, like Shoham, or one interval argument, like Allen [All84], to construct theories of truth over convex intervals.

However, union-of-convex intervals correspond to no fixed number of points - one union-of-convex interval may have four maximal convex subintervals (8 points) and another six (12 points). We cannot simply use a fixed number of point arguments. However, if we represent union-of-convex intervals directly, we utilise one temporal argument only in an assertion of truth (the union-of-convex interval), as in the convex case. This move allows us to use a standard logical language to write temporally-qualified assertions. And it doesn't make things more difficult for our application: it simplifies as well as enriches the syntactic rules.

How should the data structure *union-of-convex intervals* be implemented? Ligozat (*op. cit.*) represents such intervals as sequences of real numbers (representing points) of even length. We choose a different representation, the TUS, in Section 4. This allows two methods of reasoning: logical reasoning within a theory of union-of-convex intervals (Section 2.3), and integer calculations performable directly within the TUS itself (discussed in Section 4).

Arguments for the use of intervals to represent time are also found in [Ham71, Dow79, Hum79, All83, Lad86a, Lad86b, AH89, MSK93, vB92b]. Our proposal to use union-of-convex intervals is not shared by all of these.

2.2 Can We Use an Existing Truth-Over-Intervals Theory?

Halpern and Shoham [HS91] have proposed logics for reasoning about temporal information, using intervals as references for truth values. However, the logics have decision problems of various high asymptotic complexities. Reasoning in them is not so easy; and (for most cases, necessarily incomplete) sets of inference rules have not been proposed or justified. Further, the logics do not appear to satisfy appropriate versions of (DP) or (UP).

Shoham has proposed a reified logic [Sho87] using two points (which amounts to using convex intervals, as we have noted). An atomic predicate has the form $TRUE(x, y, p)$, or as we would write it $TRUE(i, p)$, where $i = \langle x, y \rangle$ is an Allen interval (an ordered pair of real numbers x, y with $x < y$) and p is a proposition. In addition to the usual inference rules, such a reified theory must add axioms simulating propositional reasoning in the arguments to *TRUE*, to enable the usual propositional inferences to be performed, e.g.

$$TRUE(i,p)\,\&\,TRUE(i,q)\;\Rightarrow\;TRUE(i,(p\&q))$$

Having to add all these rules is logically inelegant.

Such a reified logic is subsumed by the two-sorted first-order BTK logic. Both approaches mix the interval references in the syntax with the atomic non-temporal propositions, conforming with our desideratum for a syntactic theory. The non-reified BTK logic has one sort ranging over temporal 'objects' and the other sort over everything else. An atomic formula in this logic could be $p(I)$, where p is an atomic proposition, qualified with its temporal (interval) argument I. An atomic formula written $p(a,b)$ in first-order logic becomes a formula $p(a,b,I)$, with a,b ranging over the 'object' sort and I over the 'interval' sort. Consider the reified rule for conjunction, above. The non-reified formula corresponding to $TRUE(i,(p\&q))$ is $p(I)\,\&\,q(I)$, which also corresponds to $TRUE(i,p)\,\&\,TRUE(i,q)$. Thus, the inelegant reified rule is simply absorbed by the underlying logic in the BTK formulation (this is also true for disjunction: however, negation is problematic–see Section 5).

To use a BTK-style logic, we need:

A temporal theory: Bacchus *et al.* don't propose any. We pick union-of-convex intervals over the rationals.

Varying temporal references in some of the axioms: one should be able to infer the truth of propositions over different intervals from those they came with, else adding temporal references wouldn't give us anything over propositional logic!

In addition, a reified theory needs to add

Simulation rules as above for propositional reasoning.

Using (the theory of) union-of-convex intervals over the rationals allows us to employ a single temporal argument in either reified or BTK-type predicate symbols. Here, the phrase $I \models p$ should informally be read as asserting that proposition p is true over interval I. Note that $I \models p$ is an atomic formula, and not a metalogical assertion! It has different formal definitions in the reified and non-reified logics – see below.

Humberstone [Hum79] has considered quantifier-free reasoning over intervals. Could we simply use the syntactic translations of his semantic rules in our framework? Unfortunately not, for two reasons:

1. All inferences considered by Humberstone take place within a fixed interval reference. (The one exception is inferences concerning the negations of propositions.) Thus, suppose p and q are atomic propositions. Although we could draw the conclusion $I \models (p\&q)$ from the hypotheses $I \models p\ \&\ I \models q$ using Hunberstone's rules, we could not draw any conclusions from $I \models p\ \&\ J \models q$, even if I and J intersect;

2. The disjunction rule of [Hum79] appears to be inappropriate. In [Hum79], a disjunction $(p \vee q)$ is true over an interval I of time, written $I \models (p \vee q)$, just in case either $I \models p$ or $I \models q$. Consider your CEO and his two working jackets. In true English fashion, he wears his jackets also eating, sleeping, running and bathing, but occasionally changes one for the other. In the intro to your book you'd like to say *all this year* he wore *either* his red jacket with orange polka dots *or* his orange jacket with red polka dots, but according to Humberstone's semantics you can't assert it unless he wears precisely one of them, and not the other, all year. (Or unless you introduced some theory of disjunctive properties that compensated by the back-door.) That would be an abomination either to the employees or to your readers. Saving both groups the trouble, we conclude that it's an anomaly.

However, some of Humberstone's considerations prove useful.

2.3 Applicable Reasoning With Intervals

The logic of interval reasoning is considered in [vB92b], in particular the first-order theory of convex intervals over a dense unbounded linear order, which is proved countably categorical. An algebraic consideration of this structure, formulated in terms of the thirteen atomic relations derivable from ordering the endpoints, and an algebraic proof of countable categoricity, may be found in [LM94]. So the theory is decidable, but what about practical algorithms? It was shown how to eliminate quantifiers from an arbitrary first-order formula of this theory in [Lad88], however, that quantifier-elimination depends on an asymptotically nasty quantifier-elimination procedure for the theory of the rational numbers [CK73]. This theory and alternatives have been discussed in detail in [Kou94]. Special reasoning techniques, using methods described in [LM94], have been developed for some quantifier-free formulas [All83, LR92, LR93]. McKenzie has noted that the theory of union-of-convex intervals on the rationals is decidable via the decision procedure for S2S (which is superexponential!) [McK87]; Ligozat has shown how to perform some quantifier-free reasoning with them [Lig90, Lig91] and Morris, Shoaff and Khatib have adapted the methods of convex-interval reasoning to some special cases [MSK93].

These works all treat the intervals as objects of a mathematical structure. They show that we can fulfil some desiderata by choosing union-of-convex intervals as our temporal reference. But what about the data? Real clocks show date, hours, minutes, seconds, and can be made to show finer grained intervals of time for more money. One can represent real convex intervals of time as sequences

[year, month, day, hour, minute, second, ...]

of all possible finite lengths and it was proved in [Lad87a] that these sequences, with the appropriate interval relations between them, form a notation for the convex rational interval structure. This notation is called the BTU (*Basic Time Units*) (called TU in [Lad87a]) and forms part of the TUS (*Time Unit System*). One way of reasoning with these sequences is to perform arithmetic on the elements (since these are positive integers). This formed the basis of calculations concerning schedules and PERT charts in Kestrel Institute's 'Project Management Assistant' PMA [DJL88]. The BTU alone is not sufficient for our purposes, since we also require reasoning about the truth of propositions qualified by those intervals, and also about union-of-convex intervals as logical objects. We need the full TUS.

In the reified form, $I \models p$ is an atomic formula formed from a two-place predicate symbol which takes as arguments a term denoting an interval and a term denoting a proposition. Propositional terms are simply Boolean combinations of propositional variables (or constants); the interval terms are from the TUS, which includes sequences, as above, and further interval terms formed from these by application of the operator *periodify*, and the *conglom* and *intersect* operators (see Section 3.1).

In the non-reified BTK-language, the propositional terms are just the propositional variables (and constants). Assertions concerning Boolean combinations of these are assertions of compound Boolean formulae in the BTK-logic, and not atomic formulae involving a compound term as in the reified version. See Section 5.

3 Reasoning With Propositions over Intervals

Before discussing the principles of reasoning with propositions over intervals, we should say something about the intervals themselves. We use the terminology of [LM94] for the relations of Allen [All83]: P, M, O, S, F, D for *precedes, meets, overlaps, starts, finishes, during*, and $\breve{P}, \breve{M}, \breve{O}, \breve{S}, \breve{F}, \breve{D}$ for their converses *preceded-by, met-by*, etc. The *sum* of any two of these relations X, Y is simply $X \cup Y$, since the sum of two binary relations is the same thing as the union of those relations considered as sets of pairs.

Let $H \triangleq (M \cup O \cup S \cup D \cup F)$ and $A \triangleq (H \cup \breve{H})$. If iAj then the intervals are either completely *separate* from one another–they touch nowhere–or they're equal. A union-of-convex interval I corresponds to a set I_S of convex intervals such that $\forall i, j \in I_S : iAj$. For most purposes, I and I_S may be identified. Members of I_S are called *components* and are the maximal convex subintervals of I (called maxconsubints in [Lad86a]). Each component is a convex interval and is thus a pair of rational (or real) numbers [All83, LM94]. We recall our restriction in this paper that a union-of-convex interval has only a finite number of components. We use the terminology $C \circledin I$ to mean that $C \in I_S$.

3.1 Conglomeration and Intersection

Although most studies of union-of-convex intervals have considered only relations amongst them, there are certain natural *operations* which are useful as well.

The *conglomeration* of two intervals I and J, denoted $conglom(I, J)$ in the code in Figure 1, is the 'union' of I and J, i.e. that interval which has as components the components of I and J, except that when different components of I and J have some common subinterval or *meet*, they are merged into one component [Lad86a]. Hence $(conglom(I, J))_S$ is not in general equal to $I_S \cup J_S$, since components of I_S are not necessarily separate from components of J_S. Conglomeration is an associative and commutative operation, and so generalises to an arbitrary finite set of interval arguments. A formal definition of conglomeration is a straightforward formalisation of the intuitive definition. See Figure 1 for an algorithm for *conglom* based on the representation I_S of an interval I. Similarly, the *intersection* of two intervals, $intersect(I, J)$, is that interval which consists precisely of those subintervals which are common to both I and J (i.e. the overlapping parts of their components). See Figure 2. An quick inspection of the algorithms in Figure 1 and Figure 2 shows that they are linear in the total number of components, that is, linear in the sum of the cardinalities of I_S and J_S.

It is more convenient when dealing with intervals abstractly to use the notation $I + J$ and $I \cdot J$ for *conglom* and *intersect* respectively. Furthermore, it is appropriate: the union-of-convex intervals we consider can be embedded in various lattices of intervals, in which *conglom* and *intersect* correspond to these lattice relations.

The union-of-convex rational intervals would form a distributive lattice [Grä78, DP90] under '+' and '·', if the empty interval were included (when it is not, '·' is a partial operation).

3.2 Principles of Interval-Propositional Reasoning

Suppose we consider adding the empty interval $\langle\rangle$ and the full line $\mathbf{1} = \langle -\infty, \infty \rangle$ (we have to add the two points-at-infinity to the rationals to accomplish this). Then it's easy to show that for any interval I, $I + \langle\rangle = I$, $I \cdot \langle\rangle = \langle\rangle$, $I + \mathbf{1} = \mathbf{1}$, $I \cdot \mathbf{1} = I$. With the points-at-infinity added, one could consider including intervals with components which are half-infinite convex intervals in the set

$$Half = \{\langle -\infty, a\rangle \mid a \in \mathcal{Q}\} \ \cup \ \{\langle b, \infty\rangle \mid b \in \mathcal{Q}\}$$

where $\mathcal{Q}$ is the rational numbers. One obtains the collection of such intervals by taking the union of the set *Half* with the collection of convex intervals over the rational numbers and then taking the closure under $+$. Call these intervals the extended-rational union-of-convex intervals. It's easy to show that every extended-rational union-of-convex interval I has a complement: an interval ${}^{\sim}I$ such that $I + {}^{\sim}I = \mathbf{1}$, $I \cdot {}^{\sim}I = \langle\rangle$. We note that the extended-rational union-of-convex intervals thus form a Boolean algebra under the binary operations $+, \cdot$, unary operation ${}^{\sim}$, and constants $\langle\rangle$, $\mathbf{1}$.

We have taken the following rule of *downward-persistence* [Sho87] as true for temporally-irreflective propositions:

$$(\mathrm{DP}) : (I \models p) \,\&\, (J \subseteq I) \ \Rightarrow \ (J \models p)$$

where "$\subseteq$" denotes the *interval-containment* relation. Interval containment is defined for convex intervals as $S \cup F \cup D$: a union-of-convex I is contained in J just in case each component of I is contained in some component of J (it is straightforward to formalise this informal definition). For the interval structure we use, interval-containment is definable from '+' using the *composition principle* below. Humberstone also hypothesises downward-persistence for propositional symbols (it then follows for all positive formulae from his rules, but we have already criticised his rule for disjunction). Similarly, we have taken the following rule of limited upward-persistence to be true in the case of temporal irreflectivity:

$$(\mathrm{UP}) : (I \models p) \,\&\, (J \models p) \;\Rightarrow\; (I + J \models p)$$

The *composition principle* that

$$\forall J \subseteq I : \exists K \textit{ disjoint from } J : J + K = I$$

although false for convex intervals over the rationals (for example, take $I = \langle 1, 4\rangle$ and $J = \langle 2, 3\rangle$), is true for union-of-convex rational intervals (we may take $K = \{\langle 1, 2\rangle, \langle 3, 4\rangle\}$). One may prefer to define $\subseteq$ in terms of $+$, namely that

$$J \subseteq I \triangleq \exists K : J + K = I$$

In the presence of this definition, along with the presence of $\langle\rangle$ and $\mathbf{1}$, the composition principle is equivalent to the existence of a complement for every interval: with $I = \mathbf{1}$, the composition principle is equivalent to $\forall J : \exists K \textit{ disjoint from } J : J + K = \mathbf{1}$, which is the sentence asserting the existence of a complement K for each J; *vice versa*, assuming the existence of complements, if $J \subseteq I$ and K is therefore such that $J + K = I$, then $K' = (K \cdot {}^{\sim}J)$ is an interval disjoint from J whose sum with J is I, as required to show the composition principle (intersection is total in the presence of $\langle\rangle$).

The following rules are intuitively plausible from the hypothesis of temporal irreflectivity, and are to be found in Humberstone (we omit the Humberstone $\vee$-elim law, which we have argued is mistaken):

$$(\&\text{-intro-elim-for-propositions})\ (I \models p \,\&\, I \models q) \;\Leftrightarrow\; I \models (p \,\&\, q)$$

$$(\vee\text{-intro-for-propositions})\ (I \models p \;\vee\; I \models q) \;\Rightarrow\; I \models (p \vee q)$$

The generalised &-intro-elimin and $\vee$-intro laws follow directly from (&-intro-elim-for-propositions), ($\vee$-intro-for-propositions) and (DP). These laws therefore follow logically from, but cannot be stated in, Humberstone's formulation (since the reference intervals vary).

$$(\&\text{-I})\ (I \models p \,\&\, J \models q) \;\Rightarrow\; (I \cdot J) \models (p \,\&\, q)$$

$$(\vee\text{-I})\ (I \models p \;\vee\; J \models q) \;\Rightarrow\; (I \cdot J) \models (p \vee q)$$

In order to obtain propositional reasoning within the reified form, we add the principle that tautologies are true over any interval, and the law of non-contradiction:

$$(\mathrm{Taut})\ I \models p, \textit{ for any tautology } p$$

$$(NC1) I \neq \langle\rangle \;\Rightarrow\; I \not\models (p \,\&\, \neg p)$$

We don't yet have all required rules. For example, given $I \models p$, it is reasonable to infer that propositional consequences of p also must hold on I. Now, q is a propositional consequence of p just in case $p \rightarrow q$ is a tautology (in classical logic), so if q is a propositional consequence of p, then $I \models p \rightarrow q$ by (Taut). Our conclusion is that $I \models q$. Putting this together yields a rule that is a reified version of *Modus Ponens*:

$$\text{(MP)}\ I \models p \ \&\ I \models (p \rightarrow q) \ \Rightarrow\ I \models q$$

This rule could be generalised to varying intervals. The generalised rule follows from this version, the persistence principles and the other rules. Also note that (Taut), (MP) and (&-I) imply (&-intro-elim-for-propositions).

Thus we have rules for each propositional connective: $\&, \vee, \rightarrow, \neg$. However, only one, (&), has both introduction and elimination rules.

We have discussed these rules without paying much attention to the interval structure, except for the antecedent in the non-contradiction rule. If $\mathbf{1}$ is included, in the presence of (DP) the tautology rule may be simplified to

$$\mathbf{1} \models p, \textit{ for any tautology } p$$

Similarly, if $\langle\rangle$ is excluded, the non-contradiction rule simplifies to

$$\text{(NC)}\ I \not\models (p \ \&\ \neg p)$$

Suppose we were to include $\langle\rangle$. One question which arises is what propositions are held to be true on the empty interval? For example, may a contradiction be true? If $I \models p$ and $J \models \neg p$ then from (&-I) and (NC1) we may conclude $J \cdot I = \langle\rangle$. But if $\langle\rangle$ were an interval in those we consider, we would be able to conclude from (&-I) that $\langle\rangle \models (p\&\neg p)$. By (Taut), $I \models (p\&\neg p) \rightarrow q$ for any proposition q, thus we may conclude by (MP) that $\langle\rangle \models q$ for any proposition q. This may not in itself yield any reasons for excluding $\langle\rangle$, but allowing the empty interval makes some reasoning about convex intervals (intervals with one component) harder: adding the empty interval and the points at infinity destroy the *uniformity* property of the convex interval relations, that for any Allen relation R and convex interval i, there exists a j such that iRj. For example, consider the Allen relation *meets* and the interval $i = \langle 2, \infty\rangle$. There is no interval j such that $i\ M\ j$. Similarly there is no interval j such that $\langle\rangle\ M\ j$. This in turn destroys the validity of the Allen composition table and renders the relation algebra **IA** of intervals much more complex (indeed possibly infinite). If the algebra is infinite, path-consistency computations may no longer terminate (an example of a non-terminating path-consistency computation over an infinite algebra was given in [LM94]), and all sorts of other horrors might ensue, *prima facie* diminishing the utility of convex interval constraint reasoning. (It might also be of some relevance that the relation algebra of interval relations over a discrete order is infinite [Mad94].)

Constraint reasoning with the **IA** is eminently practical [LR96]. Since we do not wish to exclude the possibility of merging **IA**-based methods with the rules we're suggesting, it would

Axiom	Name	Acronym
$I \models p \;\&\; J \subseteq I \;\Rightarrow\; J \models p$	Downward Persistence	DP
$I \models p \;\&\; J \models p \;\Rightarrow\; (I + J) \models p$	Upward Persistence	UP
$\neg(I \models (p \;\&\; \neg p))$	Non-Contradiction	NC
$I \models p$ for every tautology p	Tautology	Taut
$I \models p \;\&\; I \models (p \rightarrow q) \;\Rightarrow\; I \models q$	Modus Ponens	MP
$I \models p \;\&\; J \models q \;\Rightarrow\; (I \cdot J) \models (p \& q)$	&-introduction	&-I
$I \models p \;\vee\; J \models q \;\Rightarrow\; (I \cdot J) \models (p \vee q)$	$\vee$-introduction	$\vee$-I

Table 1: Axioms

seem advisable to retain compatibility with **IA**. That requires that we remain with the basic lattice, and not make the move to include $\langle\rangle$, **1**, or *Half*.

We thus adopt the data structure of rational finite-union-of-convex intervals. We shall consider the rules below in the form with (NC) and (Taut), that is, without including $\langle\rangle$ or **1** in the interval structure. When writing the rules, we assume the convention that if either argument (interval or proposition) is not well-formed, then the rule is vacuous. For example, (&-I) is vacuous for any case in which $J \cdot I = \langle\rangle$.

3.3 Are the Principles Sufficient?

The axioms that appear in Table 1 are those discussed in Section 3.2. We have arrived at these axioms by including principles we have supposed to be appropriate. Some further rules in the form $\bigwedge$ *hypotheses* $\Rightarrow$ *atomic-formula* derivable from these axioms are listed in Table 2. The first section of the table includes rules which permits conclusions about the truth value of a fixed proposition p as the interval (the argument) varies. Rules for the standard logical connectives $\vee$, &, $\rightarrow$, $\neg$ are derived (even for a single-formula implication, since $p \rightarrow p$ is a tautology). The second section lists rules concerning the truth value of varying propositions with the same interval argument (i.e., $I \models p$ and $I \models q$ and their Boolean combinations). Rules for the standard binary connectives are derived. The third lists general rules for the three binary connectives. Thus, from the principles we take as axioms, it's possible to derive rules for (a) all connectives in the case for one propositional symbol but varying intervals; (b) all binary connectives for two propositional symbols in the case in which the interval is fixed (the case considered by Humberstone for propositional logic); (c) general rules (interval not fixed) for all binary connectives. Why have we not stated a completeness theorem?

Normally, *completeness* means either that (a) for any formula, either it or its negation are provable; or (b) all valid formulas in the class of intended models are syntactically provable. It's clear that (a) does not hold for the theory we are proposing: where p is a propositional symbol and I any nonempty interval, neither $I \models p$ nor $I \not\models p$ follows from our axioms. Concerning (b), the completeness theorem we would have to prove would be that our axioms

Name	Rule	Justification
DP	$I \models p \;\&\; J \subseteq I \;\Rightarrow\; J \models p$	Axiom
UP	$I \models p \;\&\; J \models p \;\Rightarrow\; (I + J) \models p$	Axiom
	$I \models p \;\vee\; J \models p \;\Rightarrow\; (I \cdot J) \models p$	DP, PrLogic
	$I \models \neg p \;\Rightarrow\; \neg(I \models p)$	NC, &-I, PrLogic
NC	$\neg(I \models (p \& \neg p))$	Axiom
Taut	$I \models p$ for every tautology p	Axiom
	$I \models (\neg p) \;\&\; J \subseteq I \;\Rightarrow\; \neg(J \models p)$	NC, DP
	$I \models p \;\&\; I \models q \;\Leftrightarrow\; I \models (p \& q)$	Taut, MP (&-I)
	$I \models p \;\vee\; I \models q \;\Rightarrow\; I \models (p \vee q)$	MP, PrLogic
MP	$I \models (p \rightarrow q) \;\&\; I \models p \;\Rightarrow\; I \models q$	Axiom
&-I	$I \models p \;\&\; J \models q \;\Rightarrow\; (I \cdot J) \models (p \& q)$	Axiom
$\vee$-P	$I \models p \;\&\; J \models q \;\Rightarrow\; (I + J) \models (p \vee q)$	Taut, MP, UP
$\vee$-I	$I \models p \;\vee\; J \models q \;\Rightarrow\; (I \cdot J) \models (p \vee q)$	Axiom
EMP	$I \models (p \rightarrow q) \;\&\; J \models p \;\Rightarrow\; (I \cdot J) \models q$	MP, DP

Table 2: 'Inference Rules' and Their Justification

enable one to derive all valid formulas of the form $I \models p$, in which I is a union-of-convex interval over the rationals and p is a Boolean formula whose atomic propositions satisfy (UP) and (DP). But what are the 'valid' formulas of this sort? We have no other characterisation of the semantics to whose definition of validity we could compare. We may simply take the validites we are proposing to be the logical closure of our axioms, thus rendering a completeness theorem trivial and useless.

We might propose a completeness theorem of the sort: all formulas of the form $I \models p$ in which I is a union-of-convex rational interval and p a Boolean combination of atomic propositions satisfying (DP) and (UP) which follow from our axioms also follow from the axioms using the rule of forward chaining only. But since propositional logic may be formulated in the Hilbert fashion using Modus Ponens only, this is trivially true and doesn't tell us anything.

A third possibility for a completeness theorem might be: consider a general lattice (or almost-lattice) of intervals. Do our rules suffice to axiomatise the valid formulas of the form $I \models p$ for p as above? Although this might be an interesting question, firstly, one must wait as in our case for an adequate alternative characterisation of the semantics of such formulas over general lattices of intervals, and we foresee numerous difficulties with that; secondly, we see no reason for it to be the case!

Finally, it would be legitimate to propose different principles, for example some that would disallow the validity of all classical tautologies on all intervals because the proposer believes that intuitionistic propositional logic, and not classical logic, is justified. It is unlikely, for example, that $I \models (p \vee \neg p)$ would be taken to hold for all intervals I and all propositions p

by a believer in intuitionistic logic. Any such discussion concerns the semantics itself, about whether certain formulae are valid or not, and not about whether certain syntactic rules yield all (previously agreed-upon) validities.

Formulating a completeness theorem is premature until the semantics are agreed and alternate syntactic ways of embodying these principles are available.

4 Interval Calculations

We have adopted the data structure of rational finite-union-of-convex intervals. We find the TUS its most convenient representation. There are other possibilities–the representation of [Lig90] uses a sequence of points $\langle a_1, \ldots, a_{2n}\rangle$ for a union-of-convex interval, in which for all i, $1 \leq i \leq (2n-1)$, $a_i < a_{i+1}$. A component is a pair of adjacent points $\langle a_i, a_{i+1}\rangle$ with i odd. A convex interval in the TUS has the form *periodify(a,b)*, where $a, b \in$ BTU (see Section 4.) A union-of-convex interval in the TUS is a set of separate components, each of which has this form. We believe that this *prima facie* more complex data representation earns its keep when representing real time, and is at least as easy to calculate with as that of Ligozat.

The TUS is based on basic units, members of BTU, which are described by sequences, whose meaning is as in Section 2.3 [Lad87a]. Convex intervals are formed from these basic units by means of the operator *convexify*, where $\mathit{convexify}(i,j) =$ *the smallest convex interval containing i and j*. It was observed in [Lad86a] that the set of names

$$\text{BTU} \;\cup\; \{\mathit{convexify}(i,j) | i,j \in \text{BTU}\}$$

with the appropriate relations denotes all the convex intervals in the TUS (i.e. only one application of *convexify* is needed with appropriate choice of arguments). We add a partial operator *periodify* to obtain a normal form for intervals for a particular context. The operator *periodify* is a way of normalising the arguments to *convexify* so that they are both BTUs of the same length. Thus if *periodify* is defined for two arguments i and j, $\mathit{convexify}(i,j) = \mathit{periodify}(i,j)$. The constraints on i, j to be arguments of *periodify* are that

1. $\mathit{length}(i) = \mathit{length}(j)$;
2. $(i\, P\, j) \;\vee\; (i\, M\, j) \;\vee\; (i = j)$

For each unit in a BTU i, for each k such that $2 \leq k \leq \mathit{length}(i)$, we define the function $\mathit{last\text{-}number\text{-}of\text{-}unit}(i,k)$ to be the last possible number of the k'th element in the sequence, given the values of the first $k-1$ elements: e.g.,

$$\mathit{last\text{-}number\text{-}of\text{-}unit}(i,2) = 12$$

$$\mathit{last\text{-}number\text{-}of\text{-}unit}(i,3) \in \{28, 29, 30, 31\}$$

$$\mathit{last\text{-}number\text{-}of\text{-}unit}(i,4) = 23$$

$$\mathit{last\text{-}number\text{-}of\text{-}unit}(i,5) = 59$$

$$\textit{last-number-of-unit}(i, 6) = 59$$

We define the use of the symbol ι as follows: $(\iota x : \phi(x))$ is any object a such that $\phi(a)$ is true, and $\langle\rangle$ if there exists none. $\iota x \in y : \phi(x)$ is short for $\iota x : x \in y \,\&\, \phi(x)$. We consider the type of sequences on S of length n to be identical to that of the finite functions with domain $\{1, 2, 3, \ldots, n\}$, and codomain S. Thus we define $\textit{Seq}(n, S) = [\{1, \ldots, n\} \rightarrow S]$, the set of all sequences of length n of elements of S. We define

$$\textit{first-unit}(i, n) \stackrel{\triangle}{=} \iota f \in \textit{Seq}(n, S) : f(k) = \ \textbf{if}\ k \leq \textit{length}(i)\ \textbf{then}\ i(k)\ \textbf{else}\ 1$$

$$\textit{last-unit}(i, n) \stackrel{\triangle}{=} \iota f \in \textit{Seq}(n, S) : f(k) = \ \textbf{if}\ k \leq \textit{length}(i) \textbf{then}\ i(k) \textbf{else}\ \textit{last-number-of-unit}(i, k)$$

where **Nat** is the natural numbers (note that $k \geq 1$ in the definition of *last-unit*). The equations axiomatising *periodify* are, for each $n \geq \textit{max}(\textit{length}(i), \textit{length}(j))$,

$$\textit{periodify}(i, j) = \textit{periodify}(\textit{first-unit}(i, n), \textit{last-unit}(i, n))$$

For fixed $n \geq \textit{max}(\textit{length}(i), \textit{length}(j))$, therefore, the term $\textit{periodify}(\textit{first-unit}(i, n), \textit{last-unit}(i, n))$ is a pseudo-normal form for TUS terms representing convex intervals. For given n, we call this the n-pseudo-normal form of a convex interval. A TUS union-of-convex interval may be represented in a pseudo-normal form as the set of its components, where each component is in n-pseudo-normal form for appropriately large n.

This pseudo-normal form is used in the pseudocode (and the code) for the algorithms for *conglom* and *intersect* in Figures 1 and 2. We illustrate the operation of these algorithms on simple examples.

4.1 Examples

We give a simple example of the conglomeration and intersection algorithms. Let

$$I_1 \ = \{ \text{ convexify}([1994, 1], [1994, 12]), [1992]\}$$

$$I_2 \ = \{ \text{ periodify}([1994, 6], [1995, 3]), [1993, 3, 4]\}$$

$$I_3 \ = \{ \text{ convexify}([1994, 8], [1995, 6])\}$$

be three union-of-convex intervals. We apply *conglom*.

Step (1) computes the set P as follows:
$P \ = \ \{\text{convexify}([1994, 1], [1994, 12]),\ [1992],\ \text{periodify}([1994, 6], [1995, 3]),$
$[1993, 3, 4],\ \text{convexify}([1994, 8], [1995, 6])\}$

Step (2) converts intervals in P into the form periodify(x, y):
$P \ = \ \{\text{periodify}([1994, 1], [1994, 12]),\ \text{periodify}([1992], [1992]),\ \text{periodify}([1994, 6],$
$[1995, 3]),\ \text{periodify}([1993, 3, 4], [1993, 3, 4]),\ \text{periodify}([1994, 8], [1995, 6])\}$

Algorithm $conglom(\{I_1, I_2, \ldots, I_k\})$

(1) Let $P = \{J | \; \exists \; I \in E : \; J \circledcirc \; I\}$, i.e. $P = \bigcup_{1 \leq i \leq k} I_i$

(2) $\forall J \in P$: convert J to form $periodify(x, y)$

(3) Let $len = max\{length(x) | \; \exists \; J \in P \; y \in \text{TU} \; J = periodify(x, y) \vee J = periodify(y, x)\}$

(4) $\forall J \in P$: convert J to form periodify($first\text{-}unit(J, len), last\text{-}unit(J, len)$)

(5) Let $EndPts = \{\langle x^\frown s\rangle | \; \exists \; J \in P : J = periodify(x, y)\}$
$\cup \; \{\langle y^\frown e\rangle | \; \exists \; J \in P : \; J = periodify(x, y)\}$

(6) L = sort($EndPts$), where sort yields lexicographic order with $s < e$

(7) $U \longleftarrow \emptyset$;
While $L \neq \langle\rangle$ do
First-unit $\longleftarrow$ $head(L)$;
$L \longleftarrow tail(L)$;
$co \longleftarrow 1$;
While $co \neq 0$ do
If $last(head(L)) = s$ then $co \longleftarrow co + 1$
else if $last(head(L)) = e$ then $co \longleftarrow co - 1$;
Last-unit $\longleftarrow$ $head(L)$;
$L \longleftarrow tail(L)$;
End-While;
$U \longleftarrow U \cup \{periodify$(First-unit,Last-unit)$\}$
End-While.

Figure 1: The conglomeration algorithm

Algorithm *intersect*$(\{I_1, I_2, \ldots, I_k\})$

(1) Let $P = \{J | \exists I \in E : J \circledin I\}$, i.e. $P = \bigcup_{1 \leq i \leq k} I_i$

(2) $\forall J \in P$: convert J to form $periodify(x, y)$

(3) Let $len = max\{length(x) | \exists J \in P, y \in \mathrm{TU} : J = periodify(x, y) \vee J = periodify(y, x)\}$

(4) $\forall J \in P$: convert J to form $periodify(\mathit{first\text{-}unit}(J, len), \mathit{last\text{-}unit}(J, len))$

(5) Let $EndPts = \{< x^\frown s\rangle | \exists J \in P : J = periodify(x, y)\}$
$\cup \{< y^\frown e\rangle | \exists J \in P : J = periodify(x, y)\}$

(6) $L = sort(EndPts)$, where sort yields lexicographic order, with $s < e$

(7) $co \longleftarrow 0$

(8) $U \longleftarrow \emptyset$;

```
While L ≠ ⟨⟩ do
    While co ≠ k − 1 and last(head(L)) = s do
        co ⟵ co + 1;
        L ⟵ tail(L);
    End-While;
    If co = k − 1 and last(head(L)) = s then
            First-unit ⟵ head(L); Last-unit ⟵ head(tail(L));
            U ⟵ U ∪ {periodify(First-unit,Last-unit)}
            L ⟵ tail(tail(L));
        else
            While (head(L) = e do
                co ⟵ co − 1;
                L ⟵ tail(L);
            End-While;
End-While
```

Figure 2: The intersection algorithm

Axiom	Name	Justification
$p(I)\ \&\ J \subseteq I\ \Rightarrow\ p(J)$	Downward Persistence (BDP)	Axiom
$p(I)\ \&\ p(J)\ \Rightarrow\ p(I+J)$	Upward Persistence (BUP)	Axiom
$\neg(p(I)\ \&\neg p(I))$	Valid	PrLogic
(Tautologies)	Valid	PrLogic
$(p(I)\ \&\ p(I) \Rightarrow q(I)) \Rightarrow q(I)$	Valid	PrLogic
$p(I)\ \&\ q(J)\ \Rightarrow\ p(I \cdot J)$	&-introduction-left (&-I-L)	BDP, PrLogic
$p(I)\ \&\ q(J)\ \Rightarrow\ q(I \cdot J)$	&-introduction-right (&-I-R)	BDP, PrLogic
$p(I)\ \vee\ q(J)\ \Rightarrow\ p(I \cdot J)\ \vee\ q(I \cdot J)$	$\vee$-introduction ($\vee$-I)	BDP, PrLogic

Table 3: Conversion of Reified Axioms to BTK Form

After step (4), we obtain:

$$\begin{aligned} P\ =\ \{&\text{periodify}([1994,1,1],[1994,12,31]),\ \text{periodify}([1992,1,1],[1992,12,31]),\\ &\text{periodify}([1994,6,1],[1995,3,31]),\ \text{periodify}([1993,3,4],[1993,3,4]),\\ &\text{periodify}([1994,8,1],[1995,6,30])\}\end{aligned}$$

Then, we compute the set $EndPts$ (step(5)) that step(6) sorts lexicographically into L:

$$\begin{aligned} L\ =\ \{&[1992,1,1,s],\ [1992,12,31,e],\ [1993,3,4,s],\ [1993,3,4,e]\\ &[1994,1,1,s],\ [1994,6,1,s],\ [1994,8,1,s],\ [1994,12,1,e],\\ &[1995,3,31,e],\ [1995,6,30,e])\}\end{aligned}$$

And finally, step(8) leads to:

$$\begin{aligned} U\ =\ (&\text{periodify}([1992,1,1],[1992,12,31]), \text{periodify}([1993,3,4],[1993,3,4]),\\ &\text{periodify}([1994,1,1],[1995,6,30]))\end{aligned}$$

The intersection of I_1, I_2, and I_3 is easy to compute using the *intersect* algorithm. After computing $P, EndPts, andL$ as in the comglomeration algorithm, step (8) leads to:

$$U\ =\ (\text{periodify}([1994,8,1],[1992,12,31])$$

5 A Reified Logic can be More Expressive

To translate into BTK-form, we must rewrite $I \models p$ as $p(I)$ for p a propositional variable (or an atomic formula in the case of quantifier-free predicate logic). Consider now the assertion $I \models (p\ \&\ q)$. No methods are proposed by Bacchus *et al.* to combine propositional variables into compound-variables (p-&-$q(I,..)$), and they argue that none is needed. One is expected to write $I \models (p\ \&\ q)$ as $p(I,..)\ \&\ q(I,..)$. (Note that $I \models (p \rightarrow q)$ translates to $p(I) \Rightarrow q(I)$.)

Name	Rule	Justification
BDP	$p(I)$ & $J \subseteq I \;\Rightarrow\; p(J)$	Axiom
BUP	$p(I)$ & $p(J) \;\Rightarrow\; p(I+J)$	Axiom
	$p(I) \;\vee\; p(J) \;\Rightarrow\; p(I \cdot J)$	BDP, PrLogic
	$\neg p(I) \;\Rightarrow\; \neg p(I)$	Tautology
	$\neg(p(I)$ & $\neg p(I))$	Tautology
	(Tautologies)	Tautologies
	$\neg p(I)$ & $J \subseteq I \;\Rightarrow\; \neg p(J)$	False!!
	$p(I)$ & $q(I) \;\Rightarrow\; p(I)$ & $q(I)$	Tautology
	$p(I) \;\vee\; q(I) \;\Rightarrow\; p(I) \;\vee\; q(I)$	Tautology
	$(p(I)$ & $p(I) \Rightarrow q(I)) \;\Rightarrow\; q(I)$	Tautology
&-I-L	$p(I)$ & $q(J) \;\Rightarrow\; p(I \cdot J)$	BDP, PrLogic
&-I-R	$p(I)$ & $q(J) \;\Rightarrow\; q(I \cdot J)$	BDP, PrLogic
$\vee$-U-I	$p(I)$ & $q(J) \;\Rightarrow\; p(I+J) \;\vee\; q(I+J)$	False!!
$\vee$-I	$p(I) \;\vee\; q(J) \;\Rightarrow\; p(I \cdot J) \;\vee\; q(I \cdot J)$	BDP, PrLogic
BEMP	$p(I) \rightarrow q(I)$ & $p(J) \;\Rightarrow\; q(I \cdot J)$	New Axiom !!

Table 4: The Rules in BTK Form

Certain rules are thus transformed into tautologies: for example, $I \models (p \;\vee\; \neg p)$ becomes $p(I, ..) \;\vee\; \neg p(I, ..)$.

Table 3 shows the reified-axioms converted into BTK-form. Three of them, (NC), (Taut) and (MP) become tautologies. Two others, (&-I) and ($\vee$-I), become three formulas in BTK-form, and all of them are easily derivable by propositional logic from (BDP). Hence, the BTK-form of the reified-axioms introduced in Section 3.2 are simply the downward and upward persistence rules (BDP) and (BUP). This might suggest that the BTK formulation is more elegant, since it consists simply of the two persistence rules that constraint the class of propositions at which we're looking. However, let's look at the reified derived rules.

Table 4 shows the BTK translation of the rules of Table 2, which are either axioms (from Table 1) or derived from axioms, as noted in the *Justification* column. The BTK-form of the derived rules yields some anomalies.

1. The rule (NC) of the reified logic turns into a false statement in the BTK logic. The reason for this may easily be seen – the reified logic distinguishes between a negation of a proposition being true over an interval I, and it not being the case that the proposition is true over I. It seems this distinction may not be made in the BTK form of the logic without introducing quantifiers.[1] Humberstone [Hum79] also discusses the roles of different interpretations of negation when one evaluates propositions on intervals

[1] Bacchus has indirectly substantiated this assertion.

(convex intervals for him).

2. The rule ($\vee$-P), which is derivable in the reified logic, corresponds to ($\vee$-U-I) in BTK-translation. ($\vee$-U-I) is false, and we don't see a way to effect a translation of ($\vee$-P) in the BTK logic without stepping outside our constraints in Section 1.2.

3. The BTK translation of (EMP) yields a new rule (BEMP) which isn't derivable from (BDP) or (BUP), even though (EMP) is derivable from (MP) (whose BTK translation is a tautology) and (DP). (BEMP) may look as though it should be derivable from propositional modus ponens and (BDP), but in fact it's not, since although one can conclude $p(I \cdot J)$ from $p(I)$ using (BDP), from $p(I) \rightarrow q(I)$ there's no rule which would enable one to conclude $p(K) \rightarrow q(K)$ for $K \subseteq I$ (and then we would use $K = I \cdot J$). The hypothesis $p(I)$ has been weakened – of course one could infer $p(I) \rightarrow q(K)$ but this doesn't help. The implication is a compound formula composed from two formulas evaluated on intervals, whereas in the reified case, it's one compound formula evaluated on a single interval, and thus (DP) is applicable.

So the BTK logic needs three rules, (BDP), (BUP) and (BEMP). Crucially, without using quantifiers it cannot easily effect a distinction which one usually wishes to make between a negation being true on an interval and it not being the case that a proposition is true on the interval; neither does it seem that the upward persistence of disjunction from a conjunction, ($\vee$-U-I), can be expressed in the simple form of an implication.

A reified logic yields discriminations that appear not to be obtainable easily with the BTK style of non-reified quantifier-free logic on union-of-convex intervals. Even supposing an equivalent form of the rules could be found, they may not have the form $\bigwedge$ *hypotheses* $\Rightarrow$ *atomic-formula* suitable for forward-chaining, and they may involve formulas other than Boolean combinations of qualified temporally-irreflective atomic predicates. We conclude that a reified logic is more suitable for evaluating propositions on union-of-convex intervals.

6 Conclusions

We considered the simple propositional logic of temporally-irreflective propositions whose truth varies with time. We argued for and used a temporal ontology of union-of-convex intervals over the rationals, represented by the TUS. We gave algorithms for calculating the operations of *conglomeration*, which is total on union-of-convex intervals, and *intersection*, over the TUS. Finally, we considered the issue of reified logic versus a BTK-style non-reified logic for evaluating propositions over union-of-convex intervals. We concluded that the non-reified logic yields anomalies, and that the reified form is preferable.

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